Advanced Higher Maths. Vectors

Reminders from Higher

Addition is commutative: $\boldsymbol{a}+\boldsymbol{b}=\boldsymbol{b}+\boldsymbol{a}$

Addition is associative: $\boldsymbol{a}+(\boldsymbol{b}+\boldsymbol{c})=(\boldsymbol{a}+\boldsymbol{b})+\boldsymbol{c}$

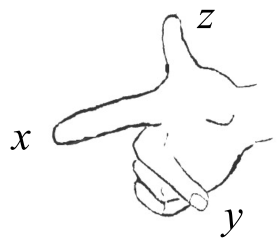
Negative of vector ***u*** **-*u*** same magnitude and direction as ***u*** but the opposite sense.

Scalar multiple If a vector ***u*** is multiplied by a scalar *k*, then
k ***u*** is a vector in same direction as ***u*** but *k* times the magnitude

Distributive Law multiplication of a vector by a scalar is distributive over addition
$(k+l)\boldsymbol{u}=k\boldsymbol{u}+l\boldsymbol{u}$ and $k(\boldsymbol{u}+\boldsymbol{v})=k\boldsymbol{u}+k\boldsymbol{v}$

3d coordinate system: can be left handed or right handed.

Right handed system; x-direction by index finger,
y direction by middle finger
z direction by thumb.



Position vector: of point A(a_1, a_2, a_3) is $\boldsymbol{a}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}$

If A and B have position vectors ***a*** and ***b*** then $\overrightarrow{AB}=\boldsymbol{b}-\boldsymbol{a}$

Mid point vector: Mid point M of the line AB is given by $\boldsymbol{m}=\frac{1}{2}(\boldsymbol{b}-\boldsymbol{a})$

Section formula: If P divides $\overrightarrow{AB}$ in the ratio $m:n$ then $\boldsymbol{p}=\frac{m\boldsymbol{b}+n\boldsymbol{a}}{m+n}$

Magnitude of a vector: $\boldsymbol{u}=\begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix}$ then the magnitude $|\boldsymbol{u}|=\sqrt{u_1^{\,2}+u_2^{\,2}+u_3^{\,2}}$

Unit vector: A vector with magnitude of 1.
A unit vector in direction of ***a*** is often written as $\boldsymbol{u}_a$
and $\boldsymbol{u}_a=\frac{1}{|a|}\boldsymbol{a}$

Unit vectors: unit vectors in x-direction, y-direction and z-direction

are denoted by: $\boldsymbol{i}=\begin{pmatrix}1\\0\\0\end{pmatrix}$, $\boldsymbol{j}=\begin{pmatrix}0\\1\\0\end{pmatrix}$ and $\boldsymbol{k}=\begin{pmatrix}0\\0\\1\end{pmatrix}$

Component form:

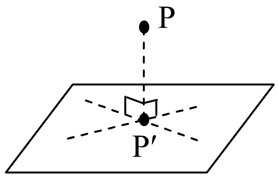
Any vector $\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ in component form

Unit vector form:

can be written in unit vector form: $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$

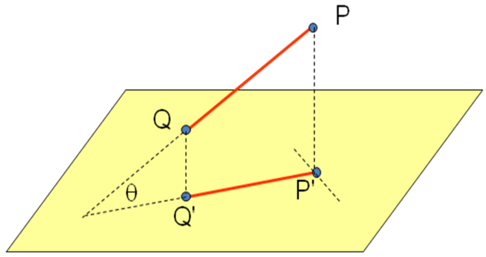
Projection of point P:

The projection of a point P on a plane is the point P′, at the foot of the perpendicular to the plane passing through P.



Projection of line PQ on a plane:

The projection of line PQ on a plane is the line P′Q′ which joins the projections of P and Q on the plane.

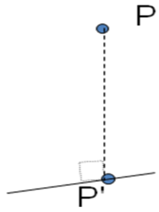


Angle θ between a line and a plane

The angle θ between a line and a plane is the angle between the line and its projection on the plane (in degrees or radians).

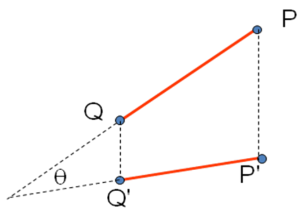
Projection of a point on a line

PP′ is the shortest distance from the point to the line (at right angles to the line)



Projection of a line segment on a line

The projection of line PQ on a line is the line P′Q′ which joins the projections of P and Q on the plane.



Scalar product:

The scalar product or dot product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is a scalar defined by: $\boldsymbol{a}.\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta$ where θ is the angle between the vectors pointing **out** of the vertex.

Note that: $|\boldsymbol{b}|\cos\theta$ is the projection of $\boldsymbol{b}$ on $\boldsymbol{a}$.

Component form of scalar product:

If $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ *and* $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$

then $\boldsymbol{a}.\boldsymbol{b} = a_1b_1 + a_2b_2 + a_3b_3$

Angle between two vectors

The angle between two vectors a and b is given by

$\cos\theta = \dfrac{\boldsymbol{a}.\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}$. If $\cos\theta < 0$ then θ is obtuse.

Properties of scalar product:	$\boldsymbol{i.i} = 1,\ \ \boldsymbol{j.j} = 1,\ \ \boldsymbol{k.k} = 1$ $\boldsymbol{i.j} = 0,\ \ \boldsymbol{j.k} = 0,\ \ \boldsymbol{i.k} = 0$ $\boldsymbol{a.a} = \lvert\boldsymbol{a}\rvert\lvert\boldsymbol{a}\rvert\cos 0 = a^2$
Perpendicular vectors	$\boldsymbol{a.b} = 0 \Rightarrow$ ***a, b*** are perpendicular Also, ***a, b*** are perpendicular $\Rightarrow \boldsymbol{a.b} = 0$
Distributive Law	Scalar product is distributive over addition $\boldsymbol{a}.(\boldsymbol{b}+\boldsymbol{c}) = \boldsymbol{a.b} + \boldsymbol{a.c}$
Commutative Law	$\boldsymbol{a.b} = \boldsymbol{b.a}$

Direction ratio:

If vector $\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ then the direction ratio is: $a_1 : a_2 : a_3$

This ratio can be used to determine the direction of the vector. Any scalar multiple of ***a*** has the same direction but not necessarily the same sense.

If vectors have equal direction ratios then they are parallel.

Direction Cosines:

If ***u*** is a **unit** vector then $\boldsymbol{u} = \begin{pmatrix} \cos\alpha \\ \cos\beta \\ \cos\gamma \end{pmatrix}$

Where α, β, and γ are the angles between the vector and *x, y, z* axes respectively.

To find direction cosines, remember to divide vector by magnitude to obtain the unit vector.

Vector Product

$\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{n}\lvert\boldsymbol{a}\rvert\lvert\boldsymbol{b}\rvert\sin\theta$ where θ is the angle between the positive directions of the vectors and the vector ***n*** is at right angles to the plane containing ***a*** and ***b***.

If ***a*** or ***b*** = 0 then ***n*** is not defined and $\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{0}$

Properties of $\boldsymbol{a} \times \boldsymbol{b}$

$\boldsymbol{a} \times \boldsymbol{b}$ is a vector in same sense and direction as ***n***

$\lvert\boldsymbol{a} \times \boldsymbol{b}\rvert = \lvert\boldsymbol{a}\rvert\lvert\boldsymbol{b}\rvert\sin\theta$ is the area of a parallelogram defined by ***a*** and ***b***

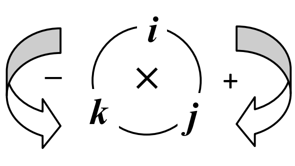
$\lvert\boldsymbol{a} \times \boldsymbol{a}\rvert = \lvert\boldsymbol{a}\rvert\lvert\boldsymbol{a}\rvert\sin 0 = 0$

$\lvert\boldsymbol{a} \times k\boldsymbol{a}\rvert = \lvert\boldsymbol{a}\rvert\lvert k\boldsymbol{a}\rvert\sin 0 = 0$. Parallel vectors have vector product of 0.

If $\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{0}$ and neither ***a*** nor ***b*** are zero, then ***a*** and ***b*** are parallel

$\boldsymbol{a} \times \boldsymbol{b} = -(\boldsymbol{b} \times \boldsymbol{a})$. The vector product is NOT commutative

$\boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k}\lvert\boldsymbol{i}\rvert\lvert\boldsymbol{j}\rvert\sin 90° = \boldsymbol{k}$, $\quad \boldsymbol{j} \times \boldsymbol{k} = \boldsymbol{i}, \quad \boldsymbol{k} \times \boldsymbol{i} = \boldsymbol{j}$



$\boldsymbol{i} \times \boldsymbol{k} = -\boldsymbol{j},\quad \boldsymbol{k} \times \boldsymbol{j} = -\boldsymbol{i},\quad \boldsymbol{j} \times \boldsymbol{i} = -\boldsymbol{k}$

$\boldsymbol{i} \times \boldsymbol{i} = \boldsymbol{j} \times \boldsymbol{j} = \boldsymbol{k} \times \boldsymbol{k} = \boldsymbol{0}$

Vector product is Distributive over addition:	$\boldsymbol{a}\times(\boldsymbol{b}+\boldsymbol{c})=\boldsymbol{a}\times\boldsymbol{b}+\boldsymbol{a}\times\boldsymbol{c}$ and $(\boldsymbol{a}+\boldsymbol{b})\times\boldsymbol{c}=\boldsymbol{a}\times\boldsymbol{c}+\boldsymbol{b}\times\boldsymbol{c}$
Component form of Vector product:	Given two vectors ***a*** and ***b*** where $\boldsymbol{a}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}$ *and* $\boldsymbol{b}=\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$ then $\boldsymbol{a}\times\boldsymbol{b}=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\a_1&a_2&a_3\\b_1&b_2&b_3\end{vmatrix}$

Scalar Triple Product:	$\boldsymbol{a}.(\boldsymbol{b}\times\boldsymbol{c})=\boldsymbol{b}.(\boldsymbol{c}\times\boldsymbol{a})=\boldsymbol{c}.(\boldsymbol{a}\times\boldsymbol{b})$ cyclically permute Also $\boldsymbol{a}.(\boldsymbol{b}\times\boldsymbol{c})$ is the volume of a parallelepiped with sides ***a, b, c.***
Properties of scalar Triple product:	$\boldsymbol{a}.(\boldsymbol{b}\times\boldsymbol{c})$ is a scalar not a vector $\boldsymbol{a}.(\boldsymbol{b}\times\boldsymbol{c})=(\boldsymbol{a}\times\boldsymbol{b}).\boldsymbol{c}$ the dot and cross are interchangeable. $\boldsymbol{a}.(\boldsymbol{b}\times\boldsymbol{c})$ is often denoted as $[\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}]$ If any of ***a, b, c*** are zero then $\boldsymbol{a}.(\boldsymbol{b}\times\boldsymbol{c})=0$
Component form of Scalar Triple	Given three vectors ***a***, ***b*** and ***c*** where $\boldsymbol{a}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix},\ \boldsymbol{b}=\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$ *and* $\boldsymbol{c}=\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}$ then $\boldsymbol{a}.(\boldsymbol{b}\times\boldsymbol{c})=\begin{vmatrix}a_1&a_2&a_3\\b_1&b_2&b_3\\c_1&c_2&c_3\end{vmatrix}$

Equation of a plane:	If $P(x,y,z)$ is a point on a plane, $\boldsymbol{a}=\begin{pmatrix}a\\b\\c\end{pmatrix}$ is a normal to the plane Then the equation of the plane is: $\boldsymbol{p}.\boldsymbol{a}=k$
Cartesian form of equation:	$ax+by+cz=k$ where *a, b, c, x, y, z* as defined above.
Find equation of plane:	Need a point $P(x,y,z)$ and normal $\boldsymbol{a}=\begin{pmatrix}a\\b\\c\end{pmatrix}$ Use: $ax+by+cz=k$ for cartesian form. (see above to find *k*)
Vector equation of a plane:	$\boldsymbol{r}=(1-t-u)\boldsymbol{a}+t\boldsymbol{b}+u\boldsymbol{c}$ This is the parametric form (p. 61/62) Where ***a, b*** and ***c*** are position vectors of points on the plane.
Angle between two planes:	Find the angle between the two normals of the planes.

Equations of a line: $\boldsymbol{r} = \boldsymbol{a} + t\boldsymbol{u}$ $\quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + t\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ this is the **vector** equation.

Where $\boldsymbol{a}$ is a point on the plane $\boldsymbol{a} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ and $\boldsymbol{u} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is

The direction vector of the line.

Parametric form: $x = x_1 + at, \quad y = y_1 + bt, \quad z = z_1 + ct$

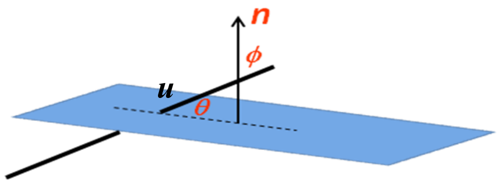
Symmetric form: $\dfrac{x - x_1}{a} = \dfrac{y - y_1}{b} = \dfrac{z - z_1}{c} = t$

Vector equation of a line from two points:

$\boldsymbol{r} = (1 - t)\boldsymbol{a} + t\boldsymbol{b}$ where $\boldsymbol{a}$ and $\boldsymbol{b}$ are points on the line.
and t is a parameter.

Angle between a line and a plane:

$\theta = 90° - \phi$



$\sin\theta = \dfrac{\boldsymbol{n}.\boldsymbol{u}}{|\boldsymbol{n}||\boldsymbol{u}|}$

NB. $\sin\theta = \sin(90° - \phi) = \cos\phi$
Where $\boldsymbol{u}$ is the line and $\boldsymbol{n}$ the normal to the plane.

Intersection of a line and a plane:

Strategy

- *Put line in parametric form*
- *Substitute x, y, x into equation of plane*
- *Solve equation to find parameter t*
- *Substitute back into x, y, z equations to find coordinate.*

Intersection of two lines: Possibilities i) parallel ii) skew iii) intersection

Strategy

- *Put line in parametric form using* t_1 *and* t_2
- *Equate expressions for x, y, x to get three equations in two unknowns.*
- *Use two of equations to find values of* t_1 *and* t_2
- *Substitute into third equation.*
  If they satisfy third equation, then you have point of intersection, if not, then the lines do not intersect. i.e. they are parallel or skew.

Intersection of two planes:

Planes must be parallel or intersect.

To find equation of intersection we need a direction vector and a point on the line.

Strategy for Point on the line

- *The line must cross the x-y plane or is parallel to it.*
- *If the line crosses it, then put $z = 0$ in each plane*
- *If parallel to $z = 0$ then use $x = 0$ or $y = 0$*
- *Solve the two equations which will give a point on the line of intersection $(x, y, 0)$, $(0, y, z)$ or $(x, 0, z)$*

Strategy for Direction Vector of the line

- *The line of intersection lies in both planes.*
- *The direction vector is therefore perpendicular to normal in both planes.*
- *So direction vector is parallel to cross product of normals.*

Intersection of three planes:

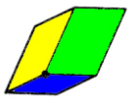
There are six possible cases.

Strategy

- *Use augmented matrix form and EROs to find solution of the three equations of the planes.*

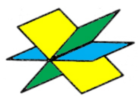
A **unique** solution gives coordinates of point of intersection.

a) A point of intersection



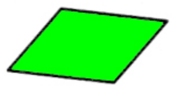
A **single redundancy** indicates a single line of intersection.

b) A single line of intersection



Two redundancies indicates a plane of intersection.

c) Plane of intersection



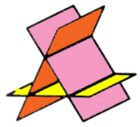
The three planes coincide
They are scalar multiples of each other.
The plane of intersection is any of the planes

One inconsistency indicates 2 or 3 lines of intersection.

d) Two lines of intersection

Two of the planes must be parallel.

e) Three lines of intersection.



Identify whether two of planes are parallel
(scalar multiples on LHS)

If two parallel planes

Solve simultaneously the non-parallel plane with each of the parallel planes in turn.

Introduce the parameter t and obtain the two equations of the lines of intersection by back substitution

If two planes are not parallel

Solve simultaneously each pair of equations of the planes.

Introduce the parameter t and obtain the three equations of the lines of intersection by back substitution

Two inconsistencies indicates no intersection.

f) No intersection.

The three planes are parallel.